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Groups with many Subgroups which are Commensurable with some Normal Subgroup

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Abstract

A subgroup H of a group G is called commensurable with a normal subgroup (*cn*) if there is $N \triangleleft G$ such that $|HN/(H \cap N)|$ is finite. We characterize generalized radical groups G which have one of the following finiteness conditions:

(A) the minimal condition on non-*cn* subgroups of G;
(B) the non-*cn* subgroups of G fall into finitely many conjugacy classes;
(C) the non-*cn* subgroups of G have finite rank.

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1 Introduction and statement of results

The investigation of the structure of groups in which all subgroups have a given property χ has been a standard in the theory of groups since the famous paper by R. Dedekind (1897) who considered normality for χ. Then many authors have considered generalization of normality in the role of χ.

In the celebrated paper [16], B.H. Neumann considered groups G in which each subgroup H is *nearly normal* (*nn* for short) in G, that is H has finite index in a normal subgroup of G, i.e. $|H^G : H|$ is finite.

Such groups turn out to be precisely the groups in which the derived subgroup G' is finite, i.e. *finite-by-abelian* groups.

A class of groups with a dual property was introduced later in [2]. A group G is said to be a CF-group if each subgroup H is *core-finite* in G (*cf* for short, or *normal-by-finite* as in [8]), that is H contains a normal subgroup of G with finite index in H, i.e. $|H : H_G|$ is finite. Clearly Tarski p-groups are CF. However, a CF-group G such that every periodic image of G is locally finite is abelian-by-finite (see [2] and [18]), that is G has an abelian subgroup with finite index. Recall that a group G is *locally finite* if every finitely generated subgroup of G is finite indeed. Abelian-by-finite CF-groups may be described in a satisfactory way using results from [4],[5] and [9]. It seems to be a still open question whether every locally graded CF-group is abelian-by-finite. Recall that a group G is *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite quotient.

To consider the above properties in a common framework one can introduce the notion of *cn-subgroup*, that is a subgroup which is commensurable with a normal subgroup. Recall that two subgroups H and K of a group G are said to be *commensurable* if $H \cap K$ has finite index in both H and K. This is an equivalence relation compatible with the intersection and will be denoted by $\sim$. In other words, we have that H *cn* G if and only if there is $N \lhd G$ such that $|HN/(H \cap N)|$ is finite. Clearly both *nn* and *cf* imply *cn*. Moreover, for finitely generated subgroups the properties *cn* and *cf* are easily seen to be equivalent (see Lemma 2.1). Also note that the intersection and the product of two *cn*-subgroups is a *cn*-subgroup (provided it is a subgroup). A group in which all subgroups are *cn* is called a CN-group. Recently, in Theorem A of [3] it has been shown that a CN-group G such that every periodic image of G is locally finite is finite-abelian-by-finite, that is G has a finite-by-abelian subgroup with finite index. However, there are soluble CN-groups which are neither abelian-by-finite nor finite-by-abelian (see [3]).

In a number of papers the structure of groups in which "many" subgroups have a given property χ has been investigated. Here we consider $\chi = cn$, where cases $\chi = nn$ and $\chi = cf$ have already been successfully treated in [6],[7] and [8]. Of course one can give different meanings to the word "many" and consider different restrictions for the non-χ subgroups. A standard one is that the set of non-χ subgroups of G satisfies the minimal condition, i.e. G has Min-χ^- (see [17]). Then one looks for group classes $\mathcal{C}$ such that a group G in $\mathcal{C}$ has Min-χ^- if and only if the following dichotomy applies:

either G *has Min on all subgroups or all subgroups have* χ*.*

For example, in [17] it was proved that this is the case when χ is normality and $\mathcal{C}$ is the class of locally graded group.

In this paper, by Theorems A, B and C we show that for the property *cn* the picture is the same as for both the properties *nn* and *cf*, for which analogous statements hold (see [6],[7],[8]).

Theorem A *Let G be a group such that every periodic image of* G *is locally finite. If G satisfies the minimal condition on subgroups which are not cn, then either G satisfies the minimal condition on all subgroups or all subgroups of* G *are cn.*

A different finiteness condition for the set of non-χ subgroups may be the imposition that it is the union of only finitely many conjugacy classes of subgroups. In fact we have:

Theorem B *Let G be a group such that every periodic image of* G *is locally finite. If G has finitely many conjugacy classes of subgroups which are not cn, then all subgroups of* G *are cn.*

Another restriction for the set of non-χ subgroups may be the requirement that it consists of subgroups with finite *(Prüfer) rank* only. Recall that a group G is said to have finite rank r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. In [6] it is shown that if in a generalized radical group in which all subgroups with infinite rank are *nn* (resp. *cf*), then all subgroups are *nn* (resp. *cf*) indeed. An analogous statement holds for *cn*-subgroups.

Theorem C *Let G be a generalized radical group with infinite rank. If each subgroup of infinite rank is cn, then all subgroups are cn.*

Notice that class of *generalized radical* groups, that is groups with an ascending series whose factors are either locally finite or locally nilpotent, is contained in the class of groups in which periodic images are locally finite.

For similar problems concerning other generalizations of normality we refer to [4], [11] and [12]. For undefined notation and basic results we refer to [10] and [15].

2 The minimal condition

We state some preliminary lemmas.

Lemma 2.1 *Let* H *be a finitely generated subgroup of group* G*. If* H *is a cn-subgroup of* G*, then* H *is a cf-subgroup.*

Proof — Let $H \sim N \lhd G$. Since N is finitely generated as well, by a well-known fact (see 1.3.7 of [15]), we have that $(N \cap H)_G$ has finite index in N, whence in H as well. $\square$

Lemma 2.2 *Let* H *be a cn-subgroup of a group* G*. If* H *is abelian-by-finite, then there exists* $A \lhd G$ *such that* $H \sim A$ *and* A *is abelian.*

Proof — If there is N such that $H \sim N \lhd G$, then such an N is abelian-by-finite as well. By a well-known fact (see [13], Lemma 21.1.4), there is a characteristic abelian subgroup A of N with finite index in N. Then A is the wished subgroup. $\square$

Lemma 2.3 *Let* $A(G)$ *be the (abelian) subgroup generated by all infinite normal cyclic subgroups of the group* G*. If* G *has Min-cn*$^-$*, then* $G/A(G)$ *is periodic.*

Proof — Let $x \in G$ with infinite order and p be any prime. Then chain

$$\langle x \rangle > \langle x^p \rangle > \ldots > \langle x^{p^i} \rangle > \ldots$$

must contain some *cn*-subgroup, say $H = \langle x^{p^i} \rangle$. By Lemma 2.1, we have $H_G = \langle x^r \rangle$ for some $r > 0$ and $x^r \in A(G)$. $\square$

Recall that a group G is called a *Chernikov group* if it has a (normal) abelian subgroup which is the direct product of finitely many Prüfer groups. Here a *Prüfer group* is a group isomorphic with the p-component of $\mathbb{Q}/\mathbb{Z}$ for some prime p.

V.P. Shunkov has proved that a *locally finite* group satisfying the minimal condition on subgroups is a Chernikov group (Theorem 5 of [19]). This result has been independently proved also by O. Kegel and B.A.F. Wehrfritz (Corollary to Theorem 1 of [14]). Moreover, for locally finite groups the minimal condition on abelian subgroups implies that the group is Chernikov indeed (see Theorem 6 of [19]). On the other hand, it is well-known that an abelian non-Chernikov group has an infinite subgroup which is the direct product of cyclic subgroups. Thus one may state the following crucial fact and a useful remark.

Lemma 2.4 *If* G *is a locally finite group which is not a Chernikov group, then* G *has an infinite abelian subgroup* A *which is direct product of groups of prime orders.*

Remark 2.5 (see Theorem 18.1 and Proposition 18.3 in [10]) *If* $A \sim B$ *are infinite abelian subgroups of a group* G *and* A *is direct product of cyclic groups, then* B *has the same property.*

Proof of Theorem A — Assume then by contradiction that G is neither CN nor Chernikov. By Lemma 2.3, G is abelian-by-locally finite. Hence each subgroup of G has the property that its locally finite images are periodic.

We claim that *a minimal-non-cn subgroup* L *of* G *is a Prüfer group*. In fact, since each subgroup H of L is a *cn*-subgrop of L, by Theorem A of [3] we have that L is finite-by-abelian-by-finite. Since L has no subgroups with finite index, L is finite-by-(divisible abelian). Moreover, as the product of two permutable *cn*-subgroups of a group is still *cn*, we have that L cannot be generated by two proper subgroups, hence L is finite-by-Prüfer. Thus L is a Chernikov group. Hence L is a Prüfer group, as claimed. Therefore *every reduced subgroup of* G *is cn*.

Assume now that G is not periodic. By Lemma 2.3, there is a non-periodic element $a \in G$ such that $\langle a \rangle \lhd G$. Note that above L centralizes $\langle a \rangle$ hence $\langle a, L \rangle = \langle a \rangle \times L$. Thus the chain

$$\langle a \rangle L > \langle a^2 \rangle L > \langle a^4 \rangle L > \ldots > \langle a^{2^n} \rangle L > \ldots$$

is strictly decreasing, so that there is a^{2^n} such that

$$K := \langle a^{2^n} \rangle L \ \mathit{cn}\ G.$$

Let $K \sim N \lhd G$. Then

$$L/(L \cap N) \simeq LN/N \leqslant KN/N$$

is finite, hence $L \leqslant N \cap K$. Thus $N \cap K = L \times \langle b \rangle$ with $b \in \langle a \rangle$. It follows that L is the finite residual of $N \cap K$. Whence L is the finite residual of $N \lhd G$

so that $L \lhd G$, contradicting the choice of L. Thus G is periodic.

By Lemma 2.4 there is an infinite abelian subgroup A of G which is the direct product of cyclic groups. Since A is reduced, $A\ \mathit{cn}\ G$ and by Lemma 2.2 and Remark 2.5 we can assume $A \lhd G$. By Theorem 2.2 in [3], there are subgroups $A_1 \leqslant A_2 \leqslant A$ of A such that both A_1

and A/A_2 are finite, and for each X such that $A_1 \leqslant X \leqslant A_2$ we have $X \lhd G$. Since the extension of a finite group by a CN-group is a CN-group, we have that G/A_1 is still a counterexample and we may assume $A_1 = 1$. By the structure of A there exist subgroups B, C of A_2 such that $BC = B \times C$, $L \cap BC = 1$ and there are strictly decreasing chain of subgroups

$$B = B_0 > \ldots > B_n > \ldots \quad \text{and} \quad C = C_0 > \ldots > C_m > \ldots$$

Then there are n, m such that LB_n and LC_m are *cn*-subgroups of G. If follows that $L = LB_n \cap LC_m$ is a *cn*-subgroup of G, the final contradiction. □

3 Finitely many conjugacy classes

When dealing with the set of conjugacy classes of subgroups of a group one may ask if this set is a poset (with respect to the relation induced by $\leqslant$). Clearly, this happens when G has the following property:

(Z) *for each* $x \in G$ *and* $H \leqslant G$ *from* $H^x \leqslant H$ *it follows* $H^x = H$.

which is trivially true when $G/Z(G)$ is periodic. Moreover, Zaicev proved that if all finitely generated subgroups of a group G satisfy the maximal condition on subgroups, then G has (Z) (see [1], Lemma 4.6.3).

Lemma 3.1 *Let* G *be a group with* (Z). *If the poset of conjugacy classes of non-cn-subgroups of* G *satisfies the minimal condition, then* G *satisfies* Min-*cn*$^-$.

Proof — In any decreasing sequence of non-*cn*-subgroups H_n, from a certain index on, all members must belong to the same conjugacy class. By property (Z), they are all equal, as requested. □

We can now state a corollary to Theorem A.

Corollary A *Let* G *be a group with (Z) and such that every periodic image of* G *is locally finite. Then* G *satisfies the minimal condition on the poset of conjugacy classes of non-cn-subgroups if and only if* G *is a Chernikov group or a CN-group.*

Let us now prove Theorem B and write, for short, that G has FMCC-*cn*$^-$ when G has finitely many conjugacy classes of non-*cn* subgroups.

Lemma 3.2 *Let* G *be a Chernikov group. If* G *has* FMCC-*cn*$^-$*, then* G *is a* CF-*group.*

Proof — It is sufficient to show that all subgroups of the finite residual R of G are G-invariant. If this is not the case, there is $x \in G$ and a Prüfer p-subgroup $P \neq P^x \leqslant R$. Thus

$$\langle P, P^x \rangle = P \times Q,$$

where Q is a Prüfer p-group. If Q_n is the subgroup of Q with order p^n we have that $(PQ_n)_{n\in\mathbb{N}}$ is an infinite family of pairwise non-isomorphic subgroups of PQ. Hence some PQ_n is *cn*. So that there is a normal subgroup

$$N \sim PQ_n \sim P.$$

Then $P \leqslant N$ and has finite index. Thus P is the finite residual of $N \lhd G$, whence $P \lhd G$, a contradiction. □

Lemma 3.3 *Let* G *be a group whose periodic homomorphic images are locally finite. If* G *has* FMCC-*cn*$^-$*, then* G *is a* CN-*group, provided each cyclic subgroup is cn.*

Proof — By Lemma 2.1, for any $x \in G$, the subgroup $\langle x \rangle$ is *cf*. Hence we have that $x^n \in A(G)$ for some n, that is $G/A(G)$ is periodic (in the same notation of Lemma 2.3). On the other hand $A(G)$ is abelian. Thus G has (Z) by above stated Zaicev result. Therefore by Lemma 3.1, G has Min-*cn*$^-$ and we are in a position to apply Theorem A and deduce that G is either CN or Chernikov. In the latter event we may apply Lemma 3.2. □

Proof of Theorem B — If G is periodic, then G has (Z), clearly. Hence G has Min-*cn*$^-$ by Lemma 3.1. Then by Theorem A and Lemma 3.2, G is a CN-group.

Let G be non-periodic and, by contradiction, not CN. Then by Lemma 3.3 G has a (infinite) cyclic subgroup $\langle x \rangle$ which is not *cn*. Let p be any prime. There exist $r, s \in \mathbb{N}$, with $r > s$, such that

$$\langle x^{p^r} \rangle \quad \text{and} \quad \langle x^{p^s} \rangle$$

are conjugate subgroups. Let $g_p \in G$ such that

$$\langle x^{p^r} \rangle^{g_p} = \langle x^{p^s} \rangle$$

and set $y := x^{p^s}$, $k := r - s$. Then

$$(y^{g_p})^{p^k} = y^{\pm 1},$$

so that

$$\langle y \rangle < \langle y^{g_p} \rangle < \langle y^{g_p^2} \rangle < \cdots$$

and the subgroup

$$Y_p := \bigcup_{n \geqslant 0} \langle y^{g_p^n} \rangle$$

is isomorphic with the additive group $\mathbb{Q}_p$ of rationals whose denominator is a power of p. Then Y_p and Y_q are not isomorphic unless $p = q$. Since G has FMCC-*cn*$^-$, there are two distinct subgroups Y_p and Y_q which are *cn*. Then the cyclic subgroup

$$Y := Y_p \cap Y_q$$

is *cn*. On the other hand, $\langle x \rangle \cap Y \neq 1$, so that $Y \sim \langle x \rangle$ and $\langle x \rangle$ is *cn* as well, a contradiction. □

4 Subgroups of infinite rank

This section is devoted to proving Theorem C. Following [6], if G is a group we shall say that a subgroup H of G has the *Neumann property* if there exists a (normal) subgroup G_0 of G such that the indices

$$|G : G_0| \quad \text{and} \quad |G_0' H : H|$$

are finite.

Clearly a normal subgroup N of G has the Neumann property if and only if the factor group G/N is finite-by-abelian-by-finite. Moreover a subgroup containing a subgroup with the Neumann property likewise has the Neumann property. Thus the class of groups in which each subgroup has the Neumann property coincides with the class of finite-by-abelian-by-finite groups.

Lemma 4.1 *Let* H, K *be subgroups of a group* G. *If* H *has the Neumann property and* $|H : (H \cap K)| < \infty$, *then* K *has the Neumann property as well. Thus the Neumann property is preserved under commensurabilty.*

PROOF — Let G_0 as above and $N := G_0'$. Then

$$|N : (N \cap K)| \leqslant |N : (N \cap K \cap H)| \leqslant |N : (N \cap H)| \cdot |H : H \cap K| < \infty.$$

The statement is proved. □

Before proving prove Theorem C, let us recall that M. De Falco, F. de Giovanni and C. Musella in [6] proved that if G is a generalized radical group in which every subgroup of infinite rank has the Neumann property, then either G has finite rank or it is finite-by-abelian-by-finite.

PROOF OF THEOREM C — We first prove that G is finite-by-abelian-by-finite. By the just quoted result, we only have to prove that every subgroup of infinite rank has the Neumann property. Let H be any subgroup of infinite rank of G. Then there is $N \triangleleft G$ such that $H \sim N$. If $H_1/N \leqslant G/N$, then H_1 has infinite rank and is therefore *cn*. Thus G/N is finite-by-abelian-by-finite, by Theorem A in [3]. By Lemma 4.1, H has the Neumann property.

It is clear that, if E is a finite normal subgroup of G and G/E is a CN-group, then G is a CN-group as well. Thus to prove that G is a CN-group, we may assume G is abelian-by-finite. So G has an abelian subgroup A of finite index. Let H be a subgroup of finite rank of G and let $H_0 = H \cap A$. There are two subgroups B and C of infinite rank of A such that

$$H_0 \cap BC = 1 \quad \text{and} \quad B \cap C = 1.$$

Then H_0B and H_0C have infinite rank, thus they are *cn*-subgroups and such is $H_0 = H_0B \cap H_0C$. Since $H \sim H_0$, also H is *cn*. □

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